

Homologically Trivial Group Actions on 4-Manifolds

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1 Introduction

In this paper we investigate pseudofree homologically trivial actions of *arbitrary* finite groups on simply connected 4-manifolds. It turns out that there are surprisingly strong restrictions on the groups and on the actions, especially when one considers the fact that any finite group can act, even freely, on some simply connected 4-manifold.

Recall that a finite group action is *free* if no nontrivial group element has a fixed point, *pseudofree* if the action is free on the complement of a discrete set, and *semifree* if it is free on the complement of the fixed point set of the whole group. An action is *locally linear* if each point has a neighborhood invariant under the isotropy group at the point on which the action is equivalent to a linear action of the isotropy group on some euclidean space. An action is said to be *homologically trivial* if the induced action on all integral homology groups is the identity.

Main Theorem. *If a finite group G acts locally linearly, pseudofreely, and homologically trivially, on a closed, simply connected 4-manifold X , with $b_2(X) \geq 3$, then G is cyclic and acts semifreely, and the fixed point set consists of $b_2(X) + 2$ isolated points.*

When the second Betti number $b_2(X) \leq 2$ other groups do arise. In the case of S^4 (corresponding to the case when $b_2 = 0$), a number of other groups do come up, since any free linear action on S^3 can be suspended to a pseudofree action, so a number of other groups arise. It turns out every group that acts pseudofreely on S^4 arises in this way. Most pseudofree actions on the complex projective plane \mathbf{CP}^2 ($b_2 = 1$) are in fact semifree, as follows from the classification of D. Wilczyński [11], [12] and of I. Hambleton and R. Lee

[7]. But there is a pseudofree, non-semifree action of $C_3 \times C_3$. Also there are standard pseudofree actions of the polyhedral groups C_n , D_{2n} , $A_4 = T$, $S_4 = O$, and $A_5 = I$ on S^2 . These non-cyclic groups give rise to pseudofree, non-semifree actions of these groups on $S^2 \times S^2$ ($b_2 = 2$). M. McCooey [8] has recently analyzed the groups that act pseudofreely on $S^2 \times S^2$. It turns out that every finite group that acts pseudofreely on $S^2 \times S^2$ also admits a linear action on \mathbf{R}^6 leaving the standard $S^2 \times S^2$ invariant, and, of course, these groups can be explicitly catalogued.

Note also that most finite cyclic groups of odd order admit such actions on any closed simply connected 4-manifold by results of A. Edmonds [6]. (There are additional restrictions on even order periodic maps originating in Rokhlin's Theorem. And there are certain cases when the order is divisible by 3 in which the group action must have a two-dimensional fixed point set.)

It remains for future work to consider actions with 2-dimensional fixed singular sets, in general.

Here is a summary of the main argument. We observe that the theorem is true for cyclic groups. We therefore attempt to proceed by induction on the order of the group. This requires separating the two conclusions (cyclic and semifree) and the consideration of some minimal non-cyclic groups. In Theorem 5.2 we show that a group acting semifreely and homologically trivially must be cyclic, provided $b_2 \geq 1$. Then in Theorem 6.1 it is shown that a homologically trivial $C_p \times C_p$ action has nonempty fixed point set, provided $b_2 \geq 2$. In Theorem 7.2 we rule out pseudofree, homologically trivial actions of nonabelian metacyclic groups when $b_2 \geq 3$. Finally the proof of the Main Theorem is completed in Theorem 8.1 where it is shown that a group acting pseudofreely and homologically trivially must in fact semifreely, provided $b_2 \geq 3$. At this last stage we apply the full induction hypothesis to consider groups in which every maximal subgroup is cyclic and acts semifreely, to reach the final conclusion, applying Theorems 6.1 and 7.2.

2 The Borel Fiberings and Spectral Sequence

We follow Bredon [3], Chapter VII, or tom Dieck [5], Chapter III, and refer to those sources for more details. Let G be a finite group. Let $E_G \rightarrow B_G$ be a universal principal G -bundle. We think of E_G as a contractible space on which G acts freely (on the right) with orbit space B_G . Although E_G and B_G are infinite dimensional they can be chosen to have finite

n -skeleton for all n . The classifying space B_G is an Eilenberg-MacLane space of type $(G, 1)$.

Now let X be a (reasonable) left G -space, say of the G -homotopy type of a finite G -CW complex. We let

$$X_G = E_G \times_G X$$

This is the bundle with base B_G and fiber X associated to the principal bundle.

One then has the spectral sequence of the fibering:

$$E_2^{i,j} = H^i(B_G; H^j(X; K)) \Rightarrow H^{i+j}(X_G; K)$$

Here K denotes any abelian coefficient group or commutative ring. It is understood that as a coefficient group $H^j(X_G; K)$ is a module over $K[G]$ and we have cohomology with twisted coefficients. The differentials

$$d_r : E_r^{i,j} \rightarrow E_r^{i+r, j-r+1}$$

have bidegree $(r, -r + 1)$.

The limit term of this spectral sequence E_∞^{**} is the graded group associated to some filtration of $H^*(X; K)$:

$$E_\infty^{i,j} = \frac{\mathcal{F}_j H^{i+j}(X_G; K)}{\mathcal{F}_{j-1} H^{i+j}(X_G; K)}$$

It is important to emphasize that $\bigoplus_{i+j=k} E_\infty^{i,j}$ is not isomorphic to $H^k(X_G)$ in general. But if K is a field, then the two vector spaces have the same dimension. And if K is a finite abelian group, then the two groups have the same cardinality, a fact we shall use.

The G -equivariant projection $E_G \times X \rightarrow X$ induces a map

$$\varphi : X_G \rightarrow X^*$$

where X^* denotes the orbit space. For $x^* = \varphi([e, x])$, one can identify $\varphi^{-1}(x^*) = B_{G_x}$.

Let $F = X^G$ denote the fixed point set of G on X . We can regard F as being embedded in both X and X^* . Also

$$X_G \supset F_G = B_G \times F$$

Let S denote the singular set for the action of G on X consisting of all points with nontrivial isotropy group. For any closed invariant subset $A \subset X$ there is a corresponding relative spectral sequence. Since the projection $\varphi : X_G \rightarrow X^*$ has point inverses of the form $\varphi(x^*) = B_{G_x}$, which is acyclic for $x \notin S$, one obtains the following result from a suitable form of the Vietoris-Begle Mapping Theorem.

Proposition 2.1. *The induced map*

$$\varphi^* : H^q(X^*, A^*; K) \rightarrow H^q(X_G, A_G; K)$$

is an isomorphism for all closed invariant $A \subset X$ containing S .

Since $H^q(X^*) = 0$ for $q > \dim X$, we have the following conclusion.

Corollary 2.2. *Inclusion induces an isomorphism $H^q(X_G) \rightarrow H^q(S_G)$ for q greater than the dimension of X .*

In particular, if G acts semifreely, then $H^q(X_G) \approx H^q(B_G \times F)$ for $q > \dim(X)$.

3 The Borel Fibering Specialized to Dimension 4

Now suppose that X is a closed, simply connected 4-manifold on which G acts homologically trivially. Then the spectral sequence has at most 3 nonzero rows ($j = 0, 2, 4$) and the only possibly nontrivial differentials are then d_3 of bidegree $(3, -2)$ and d_5 of bidegree $(5, -4)$. If, in addition, $H^*(G; K)$ vanishes in odd degrees, then the spectral sequence collapses for trivial formal reasons. A relevant example, for $K = \mathbf{Z}$, of this is the case of a 4-periodic group G , which necessarily has $H^3(G; \mathbf{Z}) = 0$. But for groups like $C_p \times C_p$, the spectral sequence need not always collapse.

A key point is that if $H^2(X; K) \neq 0$, then the spectral sequence collapses if and only if $d_3 : H^2(X; K) \rightarrow H^3(G; K)$ vanishes. (Here we identify $E_2^{02} = E_3^{02} = H^0(G; H^2(X; K)) = H^2(X; K)$ and $E_2^{30} = E_3^{30} = H^3(G; H^0(X; K)) = H^3(G; K)$.) This follows from the multiplicative structure of the spectral sequence, since the elements of $H^4(X; K)$ are products of elements in $H^2(X; K)$.

Without any further assumptions on G or X one can argue that the spectral sequence collapses for G for \mathbf{Z}_p coefficients, provided $p \neq 2$ or 3 , and $(6, |G|) = 1$, but when $p = 2$ or 3 , more care is needed. See the proof of Proposition 6.1, where we consider the case of $G = C_p \times C_p$ and $K = \mathbf{Z}_p$.

4 A Little Group Theory

Here we prove a fact about maximal proper subgroups of a general finite group, useful, as we shall see, for constructing certain proofs by induction on the order of a group. Although

this result would be well known to finite group theorists we have no easily accessible reference and so include a short proof.

Proposition 4.1. *Let G be a finite group. Then either G contains a maximal proper subgroup that is normal in G or G contains two distinct maximal subgroups that have a nontrivial intersection.*

Remark 4.2. Of course the second alternative must hold for a simple group. The example of a dihedral group D_{2p} shows that the second alternative does not always hold.

Proof. We suppose that no maximal subgroup is normal and that any two distinct maximal subgroups intersect only in the identity element and derive a contradiction.

Let $M_{i,j}$, $i = 1, \dots, r$, $j = 1, \dots, s_i$ be the full list of maximal subgroups of G , indexed so that $M_{i,j}$ and $M_{k,\ell}$ are conjugate if and only if $i = k$. We let $m_{i,j}$ denote the order of $M_{i,j}$. Since distinct maximal subgroups intersect only in the identity element, we see that $s_i = |G/M_{i,j}|$, that is G acts transitively by conjugation on $\{M_{i,j} : j = 1, \dots, s_i\}$, with isotropy group $M_{i,j}$.

We now count the total number $|G|$ of elements in G :

$$\begin{aligned} |G| - 1 &= \sum_{i=1}^r \sum_{j=1}^{s_i} m_{i,j} - 1 \\ &= \sum_{i=1}^r s_i (m_{i,j} - 1) \\ &= \sum_{i=1}^r |G| - s_i \end{aligned}$$

where $m_{i,j} = |M_{i,j}|$, which we next abbreviate to m_i . Dividing through by $|G|$ we obtain

$$1 - \frac{1}{|G|} = \sum_{i=1}^r 1 - \frac{1}{m_i}$$

But $2 \leq m_i < \frac{|G|}{2}$, so each term on the right is at least $\frac{1}{2}$. Therefore the right hand side is at least 1, while the left hand side is less than 1, unless $r = 1$. But, if $r = 1$, that is all maximal subgroups are conjugate, we have

$$1 - \frac{1}{|G|} = 1 - \frac{1}{m_1}$$

But this would mean that a maximal subgroup has the same order as the group itself, a contradiction. \square

5 Semifree Actions

If a finite group G acts pseudofreely with a nonempty fixed point set, then it acts freely on a small invariant 3-sphere around the fixed point. In particular it follows that such a group has periodic cohomology. We will show that such groups cannot act pseudofreely unless they are cyclic. We begin with the computation of such a group's integral cohomology. See, for example Brown [4], Section VI.9, including exercises 3 and 4.

Proposition 5.1. *If G is a finite group of with cohomology of period 4, then*

$$H^i(G; \mathbf{Z}) = \begin{cases} \mathbf{Z} & (i = 0) \\ G/[G, G] & (i > 0, i = 4k + 2) \\ \mathbf{Z}/|G| & (i > 0, i = 4k) \\ 0 & (otherwise) \end{cases}$$

□

Theorem 5.2. *If a finite group G acts semifreely and homologically trivially on a simply connected 4-manifold X such that $b_2(X) \geq 1$, then G is cyclic.*

Remark 5.3. Any group that acts freely (or semifreely) on S^3 acts semifreely on S^4 , by suspension. So, when $b_2 = 0$, other groups do arise.

Proof. If the fixed point set X^G contains a 2-dimensional component, then G acts effectively, preserving orientation, on the normal 2-plane to the 2-dimensional fixed point set, implying that G is indeed cyclic. Henceforth we concentrate on the case when X^G is finite.

For any $C_p < G$, the fixed sets X^G and X^{C_p} are equal, since the action is semifree, and consist of exactly $b_2(X) + 2$ points.

Since G has an isolated fixed point, G is a spherical space form group, and in particular G has periodic cohomology of period 4 (or 2 if G is in fact cyclic), concentrated in even dimensions, as described above.

Now $H^6(X_G) = H^6(X^G \times B_G)$, and both are finite. Since the cohomology of X and of G are both concentrated in even dimensions, the Borel spectral sequence collapses, and we have

$$\begin{aligned} |H^6(X_G)| &= |H^6(G)| \times |H^4(G; H^2(X))| \times |H^2(G)| \\ &= |G/[G, G]|^2 \times |G|^m \end{aligned}$$

where $m = b_2(X)$. On the other hand,

$$\begin{aligned} |H^6(X^G \times B_G)| &= |H^0(X^G) \otimes H^6(G)| \\ &= |G/[G, G]|^{m+2} \end{aligned}$$

where $m + 2$ is the number of fixed points, all of which are isolated.

It follows that $|G/[G, G]| = |G|$ and hence that $[G, G]$ is trivial. Thus G is abelian. But an abelian periodic group is cyclic. \square

6 Elementary Abelian p -Groups

We will apply a result of A. Borel [2], Chapter XII, Theorem 3.4: If an elementary abelian p -group acts in a reasonable manner on a reasonable space X such that the action is trivial on mod p homology and the mod p spectral sequence of the fibering $X_G \rightarrow B_G$ collapses, then the fixed point set X^G is nonempty and $\dim H^*(X^G; \mathbf{Z}_p) = \dim H^*(X; \mathbf{Z}_p)$. See also tom Dieck [5], Chapter III, pp. 200-201.

The following is the result we need about the action of an elementary abelian p -group of rank 2.

Proposition 6.1. *Suppose that the group $G = C_p \times C_p$ of order p^2 , where p is prime, acts effectively and homologically trivially on a closed, simply connected 4-manifold X such that $m = b_2(X) \neq 1$ if $p = 3$, and $m = b_2(X) \neq 0, 2$ if $p = 2$. Then the fixed set X^G consists of $m + 2$ isolated points.*

Proof. We fix an elementary abelian p -group G of rank 2 acting effectively and homologically trivially on a closed, simply connected 4-manifold X . The fixed point set must be finite. Otherwise G would act, preserving orientation, on a normal 2-disk to the fixed point set, implying that G is cyclic.

We analyze the Borel spectral sequence (with coefficients \mathbf{Z}_p) for the fibering

$$X \rightarrow X_G \rightarrow B_G$$

We have $E_2^{i,j} = H^i(G; H^j(X; \mathbf{Z}_p)) \Rightarrow H^{i+j}(X_G; \mathbf{Z}_p)$. According to the result of Borel mentioned above, it suffices to show that the spectral sequence collapses, that is, $E_2 = E_\infty$. For this consideration of the ring structure of the spectral sequence implies that it suffices to show that $d_3 : E_3^{0,2} \rightarrow E_3^{3,0}$, $d_3 : E_3^{0,4} \rightarrow E_3^{3,2}$, and $d_5 : E_5^{0,4} \rightarrow E_5^{5,0}$ all vanish.

Let $u \in E_3^{0,2} = H^2(X; \mathbf{Z}_p)$ be nonzero. Suppose that $u^2 \neq 0$. Of course $u^3 = 0$. Then $0 = d_3(u^3) = 3d_3(u) \cdot u^2$. It follows that $d_3(u) = 0$, provided $p \neq 3$. (We'll deal with the case $p = 3$ separately below.)

Suppose $u^2 = 0$. Then $0 = d_3(u^2) = 2d_3(u) \cdot u$, implying that $d_3(u) = 0$, provided $p \neq 2$. (We'll also deal with the case $p = 2$ separately below.) This shows that d_3 vanishes on $E_3^{0,2}$. But $E_3^{0,4} = H^4(X; \mathbf{Z}_p)$ is generated by elements of degree 2, provided $H^2(X) \neq 0$. In any case, if $v \in H^4(X; \mathbf{Z}_p)$, $0 = d_3(v^2) = 2d_3(v) \cdot v$, so that d_3 vanishes on $E_3^{0,4}$, provided $p \neq 2$.

Now we know that $E_2 = E_5$. It remains to see that d_5 vanishes. It suffices to show this for $d_5 : H^4(X; \mathbf{Z}_p) \rightarrow H^5(G; \mathbf{Z}_p)$. But this follows automatically if $H^2(X) \neq 0$, so that the latter group generates $H^4(X)$. And if $H^2(X) = 0$, it also follows for $p \neq 2$, as for d_3 .

Now consider more carefully the case that $p = 3$. We only need to show that d_3 vanishes on $E_3^{0,2}$. So let $u \in H^2(X; \mathbf{Z}_p)$. If $u^2 = 0$, then the previous argument suffices. So suppose $u^2 \neq 0 \pmod{p}$. Notice that, since $b_2(X) > 1$, there is necessarily an element $v \in H^2(X; \mathbf{Z}_p)$ such that u and v are independent and $uv = 0$. Since u and v are independent over \mathbf{Z}_p in $H^2(X; \mathbf{Z}_p)$, they are also independent in $H^2(X_G; \mathbf{Z}_p)$ over $H^*(B_G; \mathbf{Z}_p)$. But,

$$0 = d_3(uv) = du \cdot v + u \cdot dv$$

shows that u and v are dependent unless $du = 0$, as required.

Now consider more carefully the case that $p = 2$. To show that d_3 vanishes we only need to show that $d_3(u) = 0$ for nonzero $u \in H^2(X; \mathbf{Z}_2)$ such that $u^2 = 0 \pmod{2}$. Clearly then $b_2(X) \neq 1$, and the hypothesis that $b_2 \neq 2$ shows that we can find a nonzero $v \in H^2(X; \mathbf{Z}_2)$ independent of u such that $uv = 0$, i.e., $u^\perp \neq \langle u \rangle$. Then, again,

$$0 = d_3(uv) = du \cdot v + u \cdot dv$$

which shows that u and v are dependent in $H_G^2(X; \mathbf{Z}_p)$, over $H^*(B_G; \mathbf{Z}_p)$, unless $du = 0$, as required.

Since the mod p spectral sequence collapses and G acts trivially on mod p homology, the result of Borel [2] mentioned above implies that the fixed set $X^G \neq \emptyset$ and that $\dim H^*(X^G; \mathbf{Z}_p) = \dim H^*(X; \mathbf{Z}_p)$. Since we have already observed that X^G must be finite, the theorem is proved. \square

Remark 6.2. The exceptions when $p = 3$ or $p = 2$ are necessary. There is a $C_3 \times C_3$ action on \mathbf{CP}^2 with no fixed points. And there is a $C_2 \times C_2$ action on S^2 with no fixed points,

leading to a $C_2 \times C_2$ action on $S^2 \times S^2$ with no fixed points. Also it is easy to construct a linear $C_2 \times C_2$ action on S^4 with no fixed points.

Remark 6.3. An action as above cannot be semifree or pseudofree. For the existence of an isolated fixed point then implies that G acts freely on S^3 (at least up to homotopy if the action is not locally linear), so that G would have periodic cohomology. But, of course, $C_p \times C_p$ does not have periodic cohomology.

Remark 6.4. We note that a result that implies Proposition 6.1 appears in A. Assadi [1], Theorem 5.1. But the proof there is unconvincing because it makes no allowance for the counterexamples mentioned above when $b_2 \leq 2$ and $p = 2$ or 3 . With due care, however, that proof, which proceeds along lines quite different from the proof given here, can be made to work.

7 Metacyclic Groups

Next we examine the action of a nonabelian metacyclic group. In this case we do assume the action is pseudofree. If it were true that the entire group had a fixed point, then most such groups would be easily ruled out, since all but the dihedral groups have period greater than 4, while a group acting freely on a 3-sphere around a fixed point would have to have period 4. And a dihedral group cannot act freely on S^3 , by Milnor's $2p$ condition [9]. We begin with a computation of the cohomology of such a group.

In the rest of this section we suppose that G is a nonabelian extension with normal subgroup C_p and quotient group C_q , where p and q are (distinct) primes. Then there is a short exact sequence

$$1 \rightarrow C_p \rightarrow G \rightarrow C_q \rightarrow 1$$

and G has presentation of the form

$$\langle a, b : a^p = 1, b^q = a^s, bab^{-1} = a^r \rangle$$

where $r^q \equiv 1 \pmod{p}$. Since G is nonabelian, we have $r \not\equiv 1 \pmod{p}$. Moreover, the extension must in fact be split, i.e., $s = 0$. For, if b^q were nontrivial, then b would generate the whole group as a cyclic group, since we are assuming that p is prime.

We will need the cohomology of G with coefficients in the following G -modules: \mathbf{Z} , $\mathbf{Z}[G]$, $\mathbf{Z}[C_q] = \mathbf{Z}[G/C_p]$, and $\mathbf{Z}[G/C_q]$ (where we choose a representative non-normal subgroup

$C_q < G$), which correspond to the cohomology of an orbit, for each possible orbit G/H of G . Note in particular that G has periodic cohomology of period $2q$.

Lemma 7.1. *Let G be a nonabelian extension of a normal cyclic subgroup C_p of prime order p by a cyclic quotient group C_q of prime order q . Then the cohomology of G is given as follows:*

$$\begin{aligned}
H^i(G; \mathbf{Z}) &= \begin{cases} \mathbf{Z} & (i = 0) \\ \mathbf{Z}_q & (i > 0, i = 2k, k \not\equiv 0 \pmod{q}) \\ \mathbf{Z}_{pq} & (i = 2\ell q) \\ 0 & (\text{otherwise}) \end{cases} \\
H^i(G; \mathbf{Z}[G]) &= \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases} \\
H^i(G; \mathbf{Z}[G/C_q]) &= \begin{cases} \mathbf{Z} & (i = 0) \\ \mathbf{Z}_q & (i = 2k > 0) \\ 0 & (\text{otherwise}) \end{cases} \\
H^i(G; \mathbf{Z}[G/C_p]) &= \begin{cases} \mathbf{Z} & (i = 0) \\ \mathbf{Z}_p & (i = 2k > 0) \\ 0 & (\text{otherwise}) \end{cases}
\end{aligned}$$

Proof Sketch. Recall that the automorphism $C_p \rightarrow C_p$ given by $x \mapsto x^r$ induces multiplication by r^k on $H^{2k}(C_p) = \mathbf{Z}_p$. This is best seen by using the ring structure on $H^*(C_p; \mathbf{Z}_p) = \Lambda(s_1) \otimes \mathbf{Z}_p[t_2]$, where the degree 2 generator t_2 is the image of the degree 1 generator under the mod p Bockstein. Given this observation, these calculations can all be done using the Lyndon-Hochschild-Serre spectral sequence for the group extension

$$1 \rightarrow C_p \rightarrow G \rightarrow C_q \rightarrow 1$$

Further details are omitted. □

Theorem 7.2. *Suppose that a nonabelian metacyclic group G of order pq , where p and q are distinct primes, acts pseudofreely and homologically trivially on a closed, simply connected 4-manifold X , then $b_2(X) \leq 2$.*

Proof. We analyze the spectral sequence of the Borel fibering $X \rightarrow X_G \rightarrow B_G$, making use of the cohomology groups of G with coefficients in $H^0(G/H)$ as H varies over the subgroups of G .

In particular, we calculate $H^*(X_G)$ and compare the results of that calculation with a direct calculation of $H^*(S_G)$, using the fact that these two cohomology groups agree in dimensions greater than 4. where G acts on X with singular set S consisting of a finite union of orbits with isotropy groups G , $C_p \triangleleft G$, or $C_q < G$.

The spectral sequence for $H^*(X_G)$ has E_2 -term

$$E_2^{i,j} = H^i(G; H^j(X))$$

Since $H^j(X) \neq 0$ only for $j = 0, 2, 4$, and $H^i(G) \neq 0$ only for i even, we see that the spectral sequence collapses. Since the action of G on X is homologically trivial, we find that for any $n > 4$ the cohomology group $H^n(X_G)$ has a filtration with associated graded group $\mathcal{G}H^n(X_G) = (H^{n-4}(G), H^{n-2}(G)^m, H^n(G))$, where $m = b_2(X)$.

First consider the case that $n = 4q > 4$. Then $\mathcal{G}H^{4q}(X_G) = (\mathbf{Z}_{pq}, \mathbf{Z}_q^m, \mathbf{Z}_{pq})$. In particular $H^{4q}(X_G)$ is finite of order $p^2 q^{m+2}$.

On the other hand, we can compute the cohomology of S_G directly. The singular set S is a union of x_p orbits of size p (isotropy group conjugate to C_q), x_q orbits of size q (isotropy group C_p), and x_1 orbits of size 1 (isotropy group G). Each orbit of size p contributes a summand $H^{4q}(G; \mathbf{Z}[G/C_q]) = \mathbf{Z}_q$; each orbit of size q contributes $H^{4q}(G; \mathbf{Z}[G/C_p]) = \mathbf{Z}_p$; and each orbit of size 1 contributes a summand $H^{4q}(G; \mathbf{Z}) = \mathbf{Z}_{pq}$. It follows that $H^{4q}(S_G) = \mathbf{Z}_q^{x_p} \oplus \mathbf{Z}_p^{x_q} \oplus \mathbf{Z}_{pq}^{x_1}$, which has order $(pq)^{x_1} q^{x_p} p^{x_q} = p^{x_1+x_q} q^{x_1+x_p}$. Since $H^{4q}(X_G) \approx H^{4q}(S_G)$ by Corollary 2.2, we conclude that $x_1 + x_q = 2$ and $x_1 + x_p = m + 2$.

Next we consider the case that $n = 4q + 2 > 4$. Then $\mathcal{G}H^{4q+2}(X_G) = (\mathbf{Z}_q, \mathbf{Z}_{pq}^m, \mathbf{Z}_q)$. In particular $H^{4q+2}(X_G)$ is finite of order $p^m q^{m+2}$. But we may also compute directly that $H^{4q+2}(S_G) = \mathbf{Z}_q^{x_1} \oplus \mathbf{Z}_p^{x_q} \oplus \mathbf{Z}_q^{x_p}$, which has order $p^{x_q} q^{x_1+x_p}$. Therefore, $x_q = m$ and $x_1 + x_p = m + 2$.

These two sets of equations imply immediately that $m = x_q \leq x_1 + x_q = 2$. Since $m = b_2(X)$, we are done. \square

8 The Main Theorem

In this section we finish the proof of the main theorem by showing that homologically trivial, pseudofree actions are semifree, provided $b_2 \geq 3$. The argument will proceed by

induction on the order of the group. The result is easy for cyclic groups, but we also need the analyses of the preceding two sections about elementary abelian groups and metacyclic groups.

Theorem 8.1. *Suppose that a finite group G acts pseudofreely and homologically trivially on a closed, simply connected 4-manifold X such that $b_2(X) \geq 3$. Then G acts semifreely.*

Proof. For each nontrivial element $g \in G$, the fixed point set X^g consists of exactly $m + 2$ points, where $m = b_2(X)$. To see this, let $\Lambda(g)$ denote the Lefschetz number of g . Then, by the Lefschetz Fixed Point Formula we have

$$|X^g| = \chi(X^g) = \Lambda(g) = \chi(X) = b_2(X) + 2$$

Say that a subgroup H of G is *good* provided that its fixed point set X^H consists of exactly $m + 2$ points. In other words $X^H = X^h$ for all nontrivial $h \in H$. (One always has $X^H \subset X^h$.) We need to show that G itself is good. We proceed inductively toward this end. Since every cyclic subgroup is good, we may assume that every proper subgroup is good. By induction we know that such a subgroup H acts semifreely; then by Theorem 5.2 H must be a cyclic group C_k acting semifreely on X with $m + 2$ isolated fixed points, where $m = b_2(X)$.

Case 1. *There are two distinct maximal subgroups that intersect nontrivially.* Suppose that H_1 and H_2 are maximal subgroups and that $h \in H_1 \cap H_2 \setminus \{e\}$. It follows easily that the subgroup $H_1 H_2$ generated by H_1 and H_2 is also good, since H_1 , H_2 and $\langle h \rangle$ all have the same fixed point set. But maximality implies this good subgroup is G itself.

Case 2. *All maximal proper subgroups intersect trivially.* Then, by Proposition 4.1 there is a maximal proper subgroup $H < G$ that is normal in G . By maximality it follows that $G/H \approx C_q$, a cyclic group of prime order q . By induction, H acts semifreely, and so by Theorem 5.2 H is cyclic, isomorphic to C_k for some k . Thus G can be expressed as an extension

$$1 \rightarrow C_k \rightarrow G \rightarrow C_q \rightarrow 1$$

Moreover a generator of C_q acts on C_k by conjugation in G so that $x \rightarrow x^r$ for some r prime to k such that $r^q \equiv 1 \pmod{k}$. In particular, G has a presentation of the form

$$\langle a, b : a^k, b^q = a^s, bab^{-1} = a^r \rangle$$

From this it follows that there are the following four possibilities:

1. G is cyclic of order kq
2. G contains a subgroup that is a nonabelian metacyclic group of order pq with p and q distinct primes
3. G contains an elementary abelian subgroup of the form $C_p \times C_p$
4. G contains a (generalized) quaternion 2-group

Possibilities 2 and 3 are ruled out by Propositions 7.2 and 6.1, respectively. Possibility 4 is ruled out because a generalized quaternion group contains a nontrivial center, implying that all maximal subgroups intersect nontrivially, a situation which was ruled out in Case 1.

We are left with the possibility that G must be cyclic. As before, the Lefschetz Fixed Point Formula implies that G acts semifreely, completing the proof. \square

For completeness we explicitly bring the various pieces together to prove the Main Theorem.

Proof of the Main Theorem. Suppose that the finite group G acts pseudofreely, homologically trivially, locally linearly, preserving orientation, on a closed, simply connected 4-manifold X . According to Theorem 8.1 the action must be semifree. But then, by Theorem 5.2, G must be cyclic. Finally, for any non-identity element $g \in G$ we have

$$|X^G| = |X^g| = \chi(X^g) = \Lambda(g) = \chi(X) = b_2(X) + 2$$

\square

9 Concluding Remarks

One naturally wonders about removing the hypothesis of pseudofreeness in the Main Theorem. Since the 2-torus T^2 acts on standard 4-manifolds, such as connected sums of $\pm \mathbf{C}P^2$ and $S^2 \times S^2$ (see P. Orlik and F. Raymond [10]), we see that such manifolds do admit homologically trivial actions of $C_p \times C_p$, for example. Of course, the latter actions are definitely not semifree. We conjecture that these are the only sorts of groups that can act homologically trivially, when $b_2 \geq 3$. More precisely:

Conjecture 9.1. *A finite group that acts locally linearly and homologically trivially, on a closed, simply connected 4-manifold X , with $b_2(X) \geq 3$, must be abelian of rank at most 2.*

References

- [1] Amir H. Assadi, Integral representations of finite transformation groups III: simply-connected four-manifolds, *J. Pure Appl. Alg* **65** (1990), 209-233.
- [2] Armand Borel, *Seminar on Transformation Groups*, Annals of Math. Studies 46, Princeton University Press, 1960.
- [3] Glen E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.
- [4] Kenneth S. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982.
- [5] Tammo tom Dieck, *Transformation Groups*, Walter de Gruyter & Co., Berlin, 1987.
- [6] Allan L. Edmonds, Construction of group actions on four-manifolds, *Trans. Amer. Math. Soc.* **299** (1987), 155-177.
- [7] Ian Hambleton and Ronnie Lee, Finite group actions on $P^2(\mathbf{C})$, *J. Algebra* **116** (1988), 227-242.
- [8] Michael McCooey, Groups that act pseudofreely on $S^2 \times S^2$, Indiana University preprint, 1998.
- [9] John Milnor, Groups which act on S^n without fixed points, *Amer. J. Math.* **79** (1957), 623-630.
- [10] Peter Orlik and Frank Raymond, Actions of the torus on 4-manifolds I, *Trans. Amer. Math. Soc.* **152** (1970), 531-559.
- [11] Dariusz Wilczyński, Group actions on the complex projective plane, *Trans. Amer. Math. Soc.* **303**, (1987), 707-731.
- [12] Dariusz Wilczyński, Symmetries of homology complex projective planes, *Math. Zeit.* **203**, (1990), 309-319.